UK Maths Trust

Intermediate Mathematical Olympiad

Hamilton paper

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

Enquiries about the Intermediate Mathematical Olympiad should be sent to:

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1. Richard is cycling at a speed v km/h when he looks at his cycle computer to see how long it will take him to get home at his current speed. It shows a time t hours.
He cycles at this speed for 40 minutes, then instantaneously slows down by 1 km/h and checks his cycle computer; the predicted time to get home at his new speed is still t hours.
After cycling at this new speed for 45 minutes, he instantaneously slows down by another 1 km/h and checks his cycle computer; the predicted time to get home at this speed is again t hours.
How far from home was he when he first looked at the computer?

Solution

Let the initial speed be v and the distance from home he starts be d.

Then the distance from home where he started can be found in three ways from the three bits of information given.

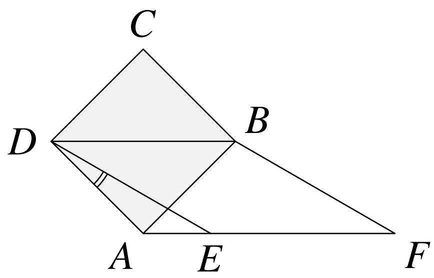
$$d = vt \tag{1}$$
$$d = \frac{2}{3}v + (v-1)t \tag{2}$$
$$d = \frac{2}{3}v + \frac{3}{4}(v-1) + t(v-2) \tag{3}$$

Equating the first two expressions gives $t = \frac{2}{3}v$.

Equating the first and third and substituting in gives $\frac{2}{3}v^2 = \frac{2}{3}v + \frac{3}{4}(v-1) + \frac{2}{3}v(v-2)$.

Solving this gives $v = 9$. Substituting in gives $t = 6$. The distance he started from home is therefore 54 km.

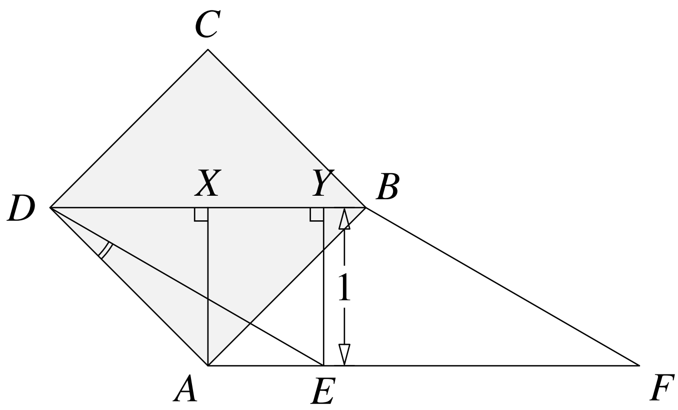
2. $ABCD$ is a square. $BDEF$ is a rhombus with A, E and F collinear. Find $\angle ADE$.



Solution

Let X be the foot of the perpendicular from A to BD and let Y be the foot of the perpendicular from E to BD.

Let BD have length 2.



The length of AX is 1 because it is half the diagonal AC, which is the same as BD. This is also the same length of EY because $AXYE$ is a rectangle.

Triangle EYD is half of an equilateral triangle (since $DE = DB = 2$), so $\angle EDY = 30°$.

Since $\angle BDA = 45°$, $\angle ADE = 45° - 30° = 15°$.

3. A large number of people arrange themselves into groups of 2, 5 or 11 people. The mean size of a group is 4. However, when each person is asked how many other people are in their group (excluding themselves), the mean of their answers is 6. Prove that the number of groups must be a multiple of 27.

SOLUTION

Let x be the number of groups of two people, let y be the number of groups of five, and let z be the number of groups of eleven. The total number of groups is $x + y + z$ and the total number of people is $2x + 5y + 11z$. We can write the mean size of a group condition as

$$2x + 5y + 11z = 4(x + y + z) \tag{1}$$

When asked how many other people are in their group:

- the $2x$ people from x groups of two reply ‘1’,
- the $5y$ people from y groups of five reply ‘4’,
- the $11z$ people from z groups of eleven reply ‘10’.

We can write the sum of their $2x + 6y + 10z$ answers as $2x + 20y + 110z$. Hence,

$$2x + 20y + 110z = 6(2x + 5y + 11z) \tag{2}$$

Expanding the brackets and collecting like terms in equations (1) and (2), we get

$$2x - y - 7z = 0 \tag{3}$$
$$10x + 10y - 44z = 0 \tag{4}$$

Substituting for y from (3) into (4) gives $5x = 19z$. Substituting for x from (3) into (4) gives $5y = 3z$.

In both cases, z must be a multiple of 5, say $5k$ for some integer k. Then $x = 19k$ and $y = 3k$. The total number of groups is $x + y + z = 19k + 3k + 5k = 27k$, so is a multiple of 27.

4. The numbers 1, 2, 3, 4 and 5 are used once each in some order substituting for the letters in the series of powers $M^{\left(A^{\left(T^{\left(H^S\right)}\right)}\right)}$. In how many of the arrangements is the units digit of the value of this expression equal to 1?

Solution

If $M = 2$, 4 or 5, the expression must be even or a multiple of 5 respectively, so cannot end in 1.

If $M = 1$, the answer will always be 1, hence end in 1. There are 24 ways of arranging the other four numbers to substitute for the remaining letters.

If $M = 3$, the answer will end in 1 precisely when the power is a multiple of 4 because the last digits of powers of 3 follow the repeating pattern of 3, 9, 7 and 1. This occurs in one of two ways.

The first is whenever $A = 4$. There are 6 ways of arranging the other three numbers to substitute for the remaining letters.

The second is when $A = 2$ and its power is larger than 1. This happens as long as $T \neq 1$. There are 2 choices for T and, for each of those, two ways of arranging the other two numbers to substitute for the remaining letters, which gives 4 ways.

The total number of arrangements is $24 + 6 + 4 = 34$.

Alternative

Consider which letter represents 1 because all the letters after that have no effect on the value of the expression.

If $M = 1$, there are 24 ways as above.

In the remaining cases, M cannot be 2, 4 or 5 as above so must be 3 and the power of 3 must be a multiple of 4.

If $A = 1$, there are no ways.

If $T = 1$, A must be 4, which gives 2 ways to arrange the values of H and S.

If $H = 1$, A must be 2 or 4 and any arrangements for T and S work after that. This gives 2 ways to arrange the values of T and S, so 4 arrangements in total.

If $S = 1$, A must be 2 or 4 and any arrangements for T and H work after that. This gives 2 ways to arrange the values of T and S, so 4 arrangements in total.

Therefore, there are $24 + 2 + 4 + 4 = 34$ arrangements.

5. The integers 1 to 100 are written on a board. Seth chooses two distinct integers from the board, b and c, and forms the quadratic equation $x^2 + bx + c = 0$. If the quadratic equation formed has integer solutions, then he erases b and c from the board; if not, the board remains unchanged.
If Seth continually repeats this process, is it possible for him to erase all the numbers from the board?

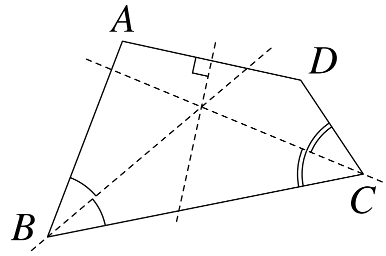
SOLUTION

We note that if $b = c+1$, the quadratic $x^2+(c+1)x+c = 0$ can be factorised into $(x+c)(x+1) = 0$ and therefore has integer solutions.

Therefore, any consecutive pair of integers can be chosen and removed from the board.

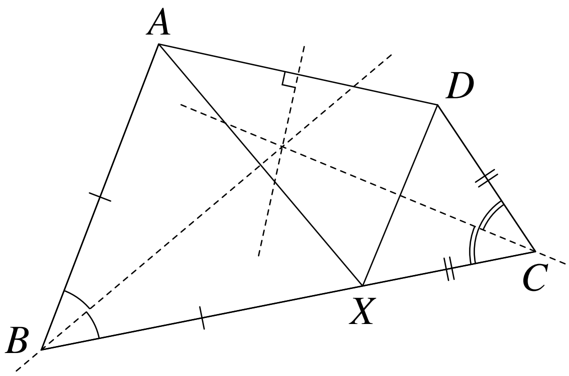
The integers from 1 to 100 can be paired up into 50 pairs of consecutive integers, each containing an odd number and the even number above it. Therefore, it is possible for Seth to erase all the numbers from the board.

6. The diagram shows a quadrilateral $ABCD$ with $AB + CD = BC$. The interior angle bisectors of $\angle B$ and $\angle C$, and the perpendicular bisector of AD, are shown as dotted lines. Prove that those three bisectors meet at a point.



Solution

Let X be the point on BC such that $BA = BX$. Then $CX = CD$.



Triangle XBA is isosceles and the angle bisector of $\angle B$ is also the perpendicular bisector of XA.

Triangle XCD is isosceles and the angle bisector of $\angle C$ is also the perpendicular bisector of XD.

The three lines are then the perpendicular bisectors of the three sides of triangle ADX. The point equidistant from A, D and X is on all three perpendicular bisectors. Therefore, the three lines intersect at this point, which is the circumcentre of triangle ADX.